

Scarring in open quantum systems

Diego Wisniacki¹ and Gabriel G. Carlo²

¹*Departamento de Física, FCEyN, UBA, Pabellón 1 Ciudad Universitaria, C1428EGA Buenos Aires, Argentina*

²*Departamento de Física, Comisión Nacional de Energía Atómica, Avenida del Libertador 8250, C1429BNP, Buenos Aires, Argentina*

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We study scarring phenomena in open quantum systems. We show numerical evidence that individual resonance eigenstates of an open quantum system present localization around unstable short periodic orbits in a similar way as their closed counterparts. The structure of eigenfunctions around these classical objects is not destroyed by the opening. This is exposed in a paradigmatic system of quantum chaos, the cat map.

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Open quantum systems are very important in many areas of physics. For example, they play a central role in the study of quantum to classical correspondence [1], microlasers [2,3], quantum dots [4], chaotic scattering [5], and more. However, there are several properties of these systems that are less known if compared to those of closed ones.

Quantum evolution in open systems is given by nonunitary matrices, whose eigenstates (resonances) are nonorthogonal and the eigenvalues are complex with modulus less than or equal to one. One of the main conjectures about the properties of the spectrum is that the mean density of resonances follows the fractal Weyl law [6]. This law predicts that the number of eigenstates that have a finite decay rate goes as $N_\gamma \propto \hbar^{-(d-1)}$, where d is a fractal dimension of the classical strange repeller. This result has been tested in some systems [7,8]. As a consequence, the majority of the eigenfunctions become degenerate with their eigenvalue modulus tending to zero as the size of the opening (the number of decay channels) relative to \hbar increases. These are the short-lived eigenstates, which cannot be associated to any classical trapped set (instead, they can be related to the trajectories that escape from the system before the Ehrenfest time). On the other hand, the number of remaining eigenstates (the long-lived ones) tends to zero. However, they contain the most relevant classical information, resembling the classical repeller. This was noticed in [9], where they were coined quantum fractal eigenstates. Moreover, this investigation was recently extended [10,11] by looking at the right and left resonances of the open baker's map. It was found that the probability density averaged for several right eigenstates is supported by the classical Cantor set (the repeller), showing self-similarity both in the q and p representation. Finally, in the more specific context of optical microcavities, the formation of long-lived scarred modes has been observed [3]. This behavior has been associated to avoided resonance crossings. Nevertheless, almost nothing else is known about the morphology of individual resonances.

We are interested in the study of quantum systems which are classically chaotic. In closed quantum chaotic systems, the morphology of the eigenfunctions has been extensively studied. One of the most important and striking properties is scarring [12]. This consists of the localization, i.e., the probability enhancement of given individual eigenfunctions along short unstable periodic orbits (POs). This effect has been discovered in the Bunimovich stadium billiard [13] and a great amount of work has been done since then [14], giving rise to what is known as "scar theory."

In this Rapid Communication we explore quantitatively the localization properties of resonances. We have studied the overlaps of wave functions highly localized on the vicinity of POs (scar functions) [15–17] with the eigenfunctions of an open quantum system in order to unveil the quantum mechanical manifestation of short POs. These values become higher than when the overlap is calculated with the eigenfunctions of the closed system. The \hbar smoothed fractal nature does not destroy structures of this kind. This effect is even greater when the area of the opening grows, thus it cannot be ascribed to a perturbative origin. We provide one with an interpretation of these results.

One of the most studied open systems corresponds to two-dimensional torus maps, where a band along the q or p direction is cut by means of a projection. The corresponding quantum dynamics is given by a nonunitary matrix $\bar{M} = PM$ (or equivalently, $\bar{M}' = MP$, which is related to \bar{M} by a time reversal operation), where M is the closed map and P is the projector on the complement of the opening. This quantum evolution is characterized by decaying eigenstates ϕ_i , whose corresponding eigenvalues z_i have complex energies. It is usual to define $|z_i|^2 = \exp(-\Gamma_i)$, where $\Gamma_i \geq 0$ is called the decay rate. We have studied a paradigmatic model of quantum chaos, the cat map, which is a linear automorphism on the two-torus generated by the 2×2 symplectic matrix \mathcal{M} , modulus 1. We have used

$$\mathcal{M} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}. \quad (1)$$

When quantizing the torus we have a finite Hilbert space of dimension $N = 1/2\pi\hbar$ and a discrete N lattice of position and momenta in the unit interval. The quantum cat map in the position representation is given by the matrix M whose elements are [18]

$$M_{kj} = \left(\frac{i}{n}\right)^{1/2} \exp\left[\frac{2\pi i}{n}(k^2 - jk + j^2)\right]. \quad (2)$$

Finally, we choose to apply the projector P after M to obtain the nonunitary matrix \bar{M} that gives the evolution of the open cat map.

The main resource that we use to investigate localization is the scar function, which not only applies to maps but also to general flows. These functions have been deeply studied in

the literature [15–17]. They are wave functions highly localized on the stable and unstable manifolds of POs, and on the energy given by a Bohr-Sommerfeld quantization condition on the trajectory. We are going to use a formulation suitable for a Poincaré surface of section, or more directly for maps of the two-torus (examples of this can be found in [16]). We define the “tube functions” for maps, $|\phi_{\text{tube}}^{\text{maps}}\rangle$, as a sum of coherent states centered at the fixed points of a given PO μ , each one having a phase [17]. Then, a dynamical average is performed, and we have the following expression for the scar function:

$$|\phi_{\text{scar}}^{\text{maps}}\rangle = \sum_{l=-T}^T e^{iS_{\mu}l/\hbar} \cos\left(\frac{\pi l}{2T}\right) M^l |\phi_{\text{tube}}^{\text{maps}}\rangle, \quad (3)$$

where T stands for the number of iterations of the map up to the Ehrenfest time $T_E = \ln \hbar / \lambda$ (λ is the Lyapunov exponent), and S_{μ} is the classical action of μ . We have used Eq. (3) to construct functions highly localized on the vicinity of the periodic points of the closed cat map. In Fig. 1(a) we can see the structure of the scar function corresponding to the PO given by $(q, p) = (0.5, 0.5)$, one of the shortest of this map, for $N=225$. The maximum probabilities correspond to the darkest regions. Panel 1(b) of the same figure displays this function in a logarithmic scale of gray, showing the way it extends along the stable and unstable manifolds of the corresponding orbit.

In the following we are going to describe the behavior of localization in the open system by means of the maximum overlaps of the scar function with its resonances. We explore different values of N and two different shapes of the projector P . For simplicity we define a map $\bar{M}_a = P_a M$, where P_a corresponds to the projection on the complement of a band parallel to the p direction of width Δq and centered at $q = q_0$. We also define the map $\bar{M}_b = P_b M$, where we now consider two symmetric bands each one having a $\Delta q/2$ width, and centered at $q = q_0$ and $q = 1 - q_0$. This is shown in the left and right insets of Fig. 3. In Figs. 1(c) and 1(d) we show the right and left eigenstates of the \bar{M}_a map that have maximum overlap with the scar function displayed in Fig. 1(a) (here $q_0 = 0.225$ and $\Delta q = 0.25$). The left resonance localizes on the unstable manifold, while the right one does it on the stable manifold. The same can be found in panel 1(e) but for the \bar{M}_b map ($q_0 = 0.1625$). We can see that the symmetric cut localizes the resonance on the stable and unstable manifolds of the trajectory. Finally, in panel 1(f) the eigenfunction of the closed cat map with maximum overlap with the scar function of panel 1(a) is shown.

First, we systematically analyze the behavior of localization as a function of \hbar . For that purpose, in Fig. 2 we show the maximum overlaps of the scar function with the right and left eigenstates of the open cat map, as a function of N (for clarity of the exposition we show the running average of these values in a window of size $\Delta N = 10$). It is evident that these values are greater for the open system, both for the right (dotted) and left (dashed) resonances. On the other hand, we can see the insets, where the order number ν_{max} of the eigenstate with maximum overlap with the scar function

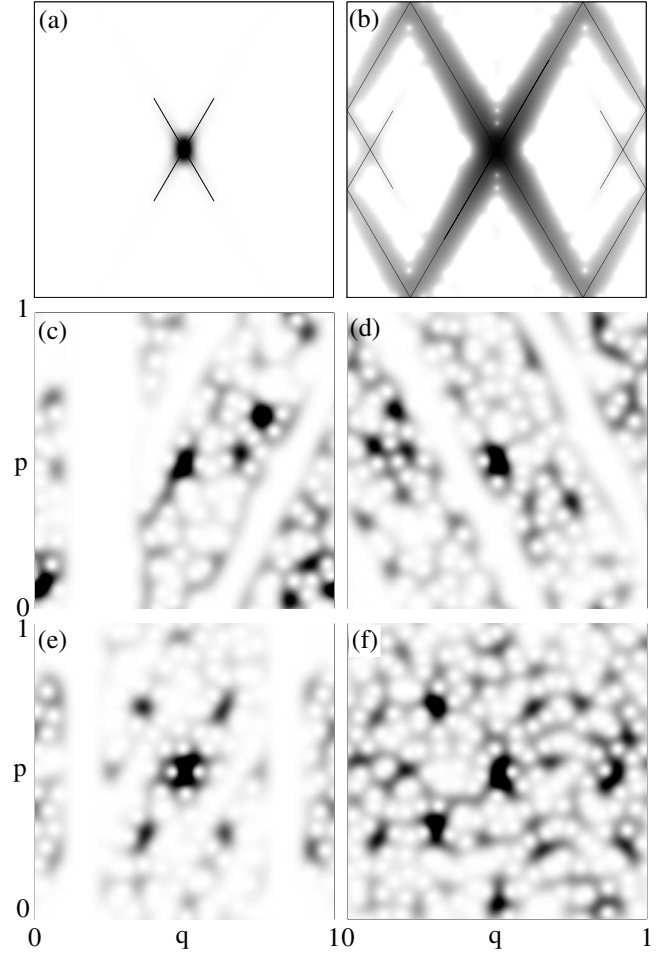


FIG. 1. (a) Scar function of the PO given by $(q, p) = (0.5, 0.5)$ for $N=225$. The horizontal axis corresponds to the position $q \in [0, 1]$ and the vertical axis to the momentum $p \in [0, 1]$ coordinate. (b) Logarithmic version of (a). Black lines correspond to the stable and unstable manifolds of the orbit. (c) Right eigenvector of the \bar{M}_a map, which has the maximum overlap with the scar function shown in (a). (d) Same as previous panel but for the left eigenstate. (e) Same as (c) in the \bar{M}_b map case. (f) Eigenfunction of the closed system having maximum overlap with the scar function of panel (a). See the text for more details.

is plotted vs N (they were ordered in ascending eigenvalue modulus). It is clear that the maximum overlap corresponds to resonances with the smaller decay rates (larger eigenvalue moduli). This guarantees that we are looking at wave functions which have a support on the classical repeller and do not belong to the null subspace. In Fig. 2(a) we can see the results for the \bar{M}_a map, while in 2(b) they correspond to \bar{M}_b , where q_0 and Δq values are taken the same as in the particular case of Fig. 1. The difference between the two maps turns out to be very important. In fact, the greater overlaps were obtained when opening in a symmetric fashion rather than with a single strip. Finally, we mention that the overlaps of the scar function with the left or the right resonances of the open map differ. These eigenstates are supported by different trapped classical sets, so in principle, there is no reason for them to coincide. Anyway, we think that the detailed expla-

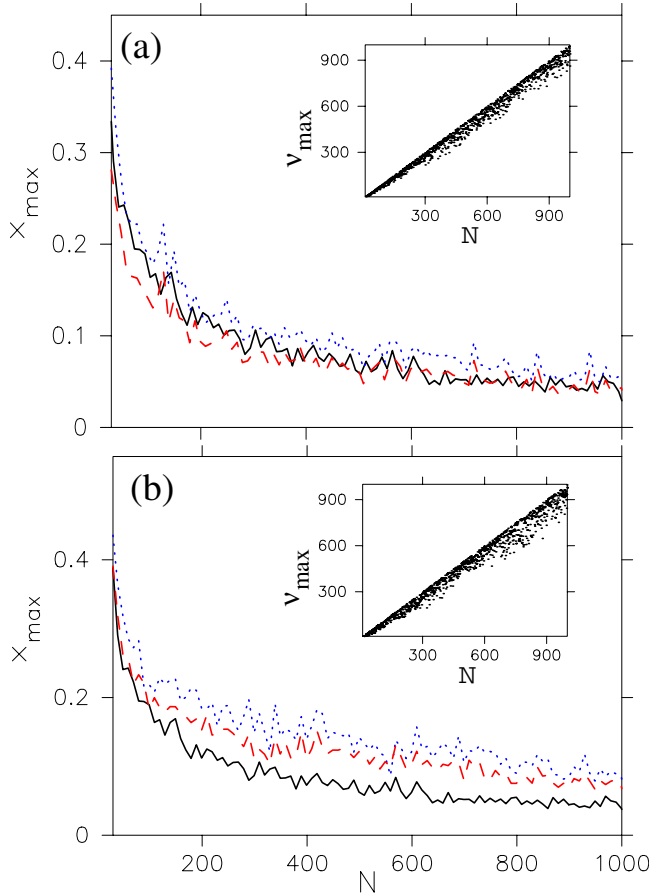


FIG. 2. (Color online) Maximum overlap x_{\max} of the scar function with the eigenstates of the open cat map as a function of N (running average in a window $\Delta N=10$). In the insets we show the order number ν_{\max} in ascending eigenvalue modulus of the right resonance with maximum overlap, as a function of N (for the left resonance, results are similar). Panel (a) corresponds to the \bar{M}_a map, while the \bar{M}_b map results are shown in (b). In all panels, the solid black lines correspond to the maximum overlap for the closed map, and the dotted and dashed lines correspond to the maximum overlaps with the right and left resonances of the open maps, respectively.

nation of this difference is an interesting open problem.

But then a natural question arises: how does the relationship between the shape and the size of the projection influence the intensity of scarring? For instance, this is relevant if we want to obtain highly localized resonances with the minimum amount of losses. This happens in many applications, the cases of two-dimensional billiards that can be used as optical microcavities for lasers or that can be attached to perfect leads, being some examples. To answer this we have further investigated the behavior of localization by fixing the \hbar value, and studying how the width of the opening influences it for both P_a and P_b operators. The results are shown in Fig. 3, where we display the average of x_{\max} taken from $N=350$ to $N=360$, as a function of the width of the opening Δq . In all cases we take $q_0=0.225$ for \bar{M}_a and $q_0=0.1625$ for \bar{M}_b . The overlaps were calculated with the right (dotted) and left (dashed) eigenstates. The lower curves correspond to \bar{M}_a ,

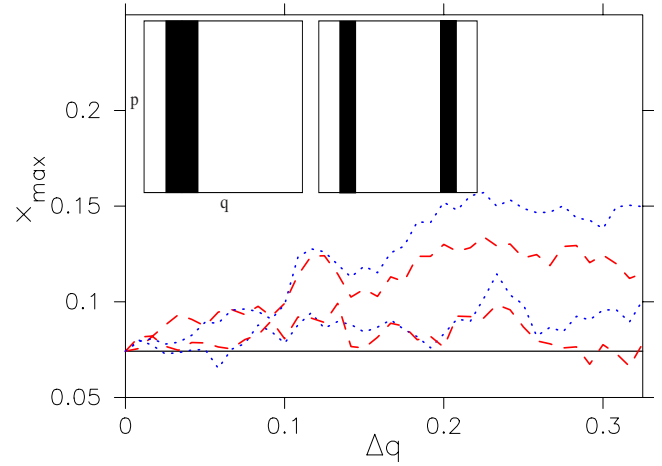


FIG. 3. (Color online) Maximum overlap x_{\max} (average from $N=350$ to $N=360$) of the scar function with the right (dotted) and left (dashed) eigenstates of the open cat map as a function of the size of the opening Δq . The lower curves correspond to \bar{M}_a , while the upper ones correspond to \bar{M}_b . The solid horizontal line stands for the value corresponding to the closed cat map. Left and right insets illustrate the projectors P_a and P_b .

while the upper ones correspond to \bar{M}_b . We have found that not only the overlap in general increases with the size of the opening, but also that this effect is greater due to the symmetrization.

But this seemingly greater scarring effect in open systems should be interpreted in the proper context. In order to do this we will analyze the weight that these long-lived resonances have in the whole spectrum, and relate it with typical time scales of the system. This is given by a connection between the fractal Weyl law and the Ehrenfest time $T_0 = \ln(O)/\lambda$ (with O the number of open channels, and $\lambda = 1.31$ in our case), first obtained in [7]. There, it was found

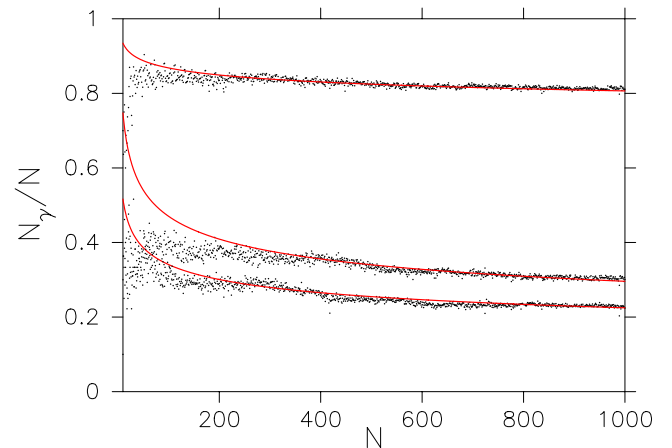


FIG. 4. (Color online) Fraction of eigenstates N_γ/N whose decay rate Γ is smaller than $\Gamma_f=0.71$, as a function of N . The lines correspond to the theoretical expression $O^{[-1/(\lambda T_0)]}$ (see the text for details). The upper curve corresponds to \bar{M}_a with an opening defined by $q_0=0.125$ and $\Delta q=0.05$, the middle one by $q_0=0.225$ and $\Delta q=0.25$, and finally, the lower one corresponds to \bar{M}_b with $q_0=0.1625$ and $\Delta q=0.25$.

that the fraction of resonances with decay rate Γ smaller than a fixed value $\Gamma_f < 1/T_0$ behaves like $N_f/N = \exp(-T_0/T_d) = O^{-1/(\lambda T_d)}$, where $T_d = N/O$ is the so-called “dwell time” (T_d large). We have numerically confirmed the validity of this prediction in our system by fitting the data with the expression $N_f/N = aN^{-b}$. We show three cases in Fig. 4, where we have taken $\Gamma_f = 0.71$ in all of them. The upper curve corresponds to \bar{M}_a with an opening defined by $q_0 = 0.125$ and $\Delta q = 0.05$, showing a fitted $b_f = 0.032$ that agrees with the theoretical $b_{th} = 0.038$. The middle curve corresponds to $q_0 = 0.225$ and $\Delta q = 0.25$ with $b_f = 0.181$, and the lower one corresponds to \bar{M}_b with $q_0 = 0.1625$ and the same Δq with $b_f = 0.2$, being $b_{th} = 0.191$ for both. In all cases this fraction goes to zero, leaving a small amount of classically meaningful eigenstates. This directly implies a persistent localization effect on the few remaining “fractal eigenfunctions.” However, these greater overlaps could correspond to a normalization difference with respect to the closed system. In fact, the effective size of the Hilbert space of the open system is smaller. Then, if we can go further and claim that these results mean a true enhancement of scarring, remains to be determined [19].

In summary, we have found that there is a greater overlap

of the scar functions with the resonances of an open system compared to the closed one. The fractal structure of the eigenstates has been widely studied, motivating their denomination as “quantum fractal eigenstates.” However, this significant alteration of the morphology of the eigenfunctions with respect to the analog closed system does not destroy the localization around POs. We think that this is due to the fact that the pruning of orbits that escape through the openings before the Ehrenfest time leaves parts of the stable and unstable manifolds. These remaining parts are enough to support the quantum structure associated to the scar function. However, the way they interfere in order to construct the same object as the smooth manifolds of the closed system remains unknown. Also, the scarring grows with the size of the opening ruling out any perturbative explanation for this. Moreover, this phenomenon is persistent, in the sense that it survives in the vanishing fraction of long-lived resonances as N grows. In future studies [19] we will focus on these open questions.

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